Unsteady Flow in Cavitating Turbopumps

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Unsteady flow in a cavitating axial inducer pump is analyzed with the help of a simple two-dimensional cascade model. This problem was motivated by a desire to study the effect of unsteady cavitation on the so-called POGO instability in the operation of liquid rocket engines. Here, an important feature is a closed loop coupling between several different modes of oscillation, one of which is due to the basic unsteady characteristics of the cavitation itself. The approaching and leaving flow velocities up- and downstream of the inducer oscillate, and the cavity-blade system participates dynamically with the basic pulsating flow. In the present work, attention is focused on finding a transfer matrix that relates the set of upstream variables to those downstream. This quantity, which is essentially equivalent to cavitation compliance in the quasi-static analyses, is found to be complex and frequency dependent. It represents the primary effect of the fluctuating cavity in the system. The analysis is based on a linearized free streamline theory.

Introduction

The problem of unsteady internal cavitating flows such as frequently observed in a pump or a turbine has drawn renewed attention recently in connection with its role played in the so-called POGO instability during the operation of liquid propellant rockets. This kind of system instability typically arises during the booster stage of flight, and is attributed to a feedback coupling between the cavitating feed pump, the supporting structure, and the motion of the propellant in the feed lines to the pump. Naturally, because of this rather complicated situation due to the coupling effect, the problem has remained somewhat unexplored in its full essence. For instance, although it is generally believed that the unsteady behavior of the cavitation itself is of essential importance to the problem, it is only very recently that an attempt was made to incorporate this into the problem [1, 2].[1]

Some of the earlier pioneering works on this subject by Rubin [3, 4], Fashbaugh and Streeter [5], and Ghahremani [6] adopted the following typical assumptions in order to make the problem more treatable:

First, a passive "compliance" is attributed to the presence of cavitation in the pump, that is, the presence of cavitation is visualized as acting like a pressurized reservoir, and numerical values for the pump cavitation compliance are determined from dynamic experiments, or test stand firings, to make the observed frequencies match the theroetical ones;

Secondly, the behavior of the pump during the unsteady motions of the transient flow is assumed to be quasisteady, namely, the change in pump performance parameters with flow rate and inlet pressure is assumed to be the same as that corresponding to steady state operation.

Brennen and Acosta [7] used this quasisteady line of approach in analyzing cavitating cascade. This type of analysis may be a correct one for a first step, and if the frequencies of oscillation were sufficiently low, the quasisteady representation of the pump performance might prove to be satisfactory. On the other hand, it should be pointed out that the inherent unsteadiness of the cavitation is not taken into account in these works, in spite of the fact that this fluctuating behavior of the cavity is an important source of system instability.

The unsteady characteristics of a cavity flow have been considered in a recent study by Kim and Acosta [1] using a simple dynamic model of a base-cavitating wedge in a tunnel, and it was concluded that the quasisteady analysis was not adequate for a wide range of frequencies of oscillation. In the present work, we will treat an unsteady flow through a cavitating axial inducer pump using the linearized free streamline theory. The inducer can be represented by a two-dimensional cascade model.

Formulation of the Problem

Consider a two-dimensional unsteady flow past a cavitating inducer cascade as sketched in Fig. 1. Let us assume the cascade blades to be flat plates, semi-infinite in length, with a stagger angle of γ. The steady angle of incidence α will be supposed to be small; the cavity is assumed to be slender so that it may be represented as a slit along the vane. The flow is approximated to be incompressible, inviscid, and irrotational. Far upstream, there exists only an axial velocity fluctuation approaching the inducer denoted by $\tilde{N}_1 e^{i\omega t}$ and no tangential velocity fluctuation. Far downstream, the flow must be parallel to the vanes, and for

[1]Numbers in brackets designate References at end of paper.

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this reason both the normal or axial ($\tilde{N}_2 e^{j\omega t}$) and the tangential ($\tilde{T}_2 e^{j\omega t}$) fluctuations will prevail. It is also assumed that the whole flow system oscillates at a single frequency ω.

Now, in the absence of the body-cavity system, elementary dynamic principles may be applied to calculate the difference in fluctuating pressures between any two remote points in the flow in terms of the pulsating velocity components. The primary effect of inserting the unsteady cavity-blade system (which interferes dynamically with the flow) is then to alter the pressure at these remote points by additional amounts $\tilde{P}_1 e^{j\omega t}$ and $\tilde{P}_2 e^{j\omega t}$ respectively for the given velocity disturbances.

Our objective here is to determine these residuary pressures $\tilde{P}_1$ and $\tilde{P}_2$ in terms of the fluctuating velocity components for given flow geometry and frequency so that ultimately we may be able to relate the four quantities ($\tilde{P}_1$, $\tilde{P}_2$) and ($\tilde{N}_1$, $\tilde{N}_2$) by

$$\begin{pmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{pmatrix} = cM \begin{pmatrix} \tilde{N}_1 \\ \tilde{N}_2 \end{pmatrix} \qquad (1)$$

where M is a 2×2 matrix and c is a factor that nondimensionalizes M. M is equivalent to a transfer matrix that relates the upstream conditions to the downstream ones.

The cavity terminus will be assumed fixed. This artificial assumption will make the unsteady and the steady flow problems separable, and is justifiable only because we are interested in the overall volume fluctuation due to the cavity and not in the local behavior of the cavity length variation.

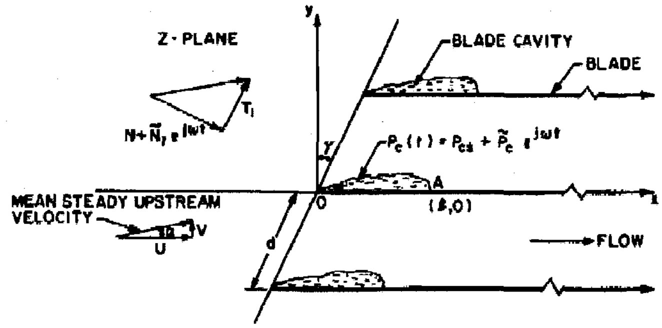

Fig. 1 Sketch of unsteady flow through cavitating cascade

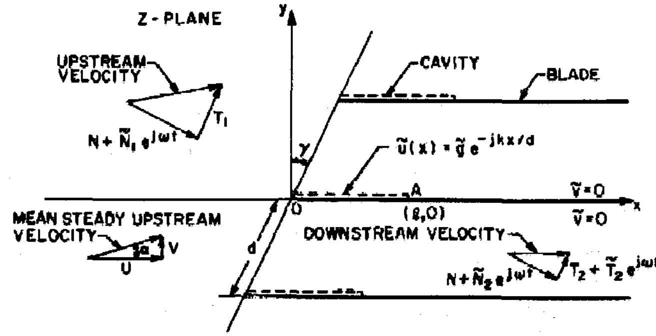

Fig. 2 Linearized z-plane with boundary conditions

Solution of the Problem

Let us denote the velocity components in the x- and the y-directions by $U + u$ and v, respectively, where $|u|/U$, $|v|/U << 1$.

On the cavity, the linearized Euler's equation in the x-direction gives

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} = 0.$$

Writing the perturbation velocity $u = u_{cs} + \tilde{u}_c(x)e^{j\omega t}$ on the cavity, one finds $\tilde{u}_c(x) = \tilde{g}e^{-jkx/d}$. Here, $k \equiv \omega d/U$ is reduced frequency and g is a constant to be determined from the solution. Along the vanes outside the cavity, $v = 0$. The linearized z-plane is shown in Fig. 2 with appropriate boundary conditions.

The complex perturbation velocity $w = u - iv$ is an analytic function of $z = x + iy$ at each instant by incompressibility and irrotationality.

To find the solution for w, it is convenient to map the linearized z-plane into the upper half of an auxiliary plane by the following transformation [8, 9]:

$$z = \frac{d}{2\pi}\left[e^{-i\gamma} \ln(1 - \zeta/\zeta_1) + e^{i\gamma} \ln(1 - \zeta/\bar{\zeta}_1) \right] \qquad (2)$$

in which the branch point $\zeta_1 = \sqrt{d}\, e^{i(\pi/2 - \gamma)}$ corresponds to upstream infinity in the z-plane. Writing $w = w_s(\zeta) + \tilde{w}(\zeta)e^{j\omega t}$

Nomenclature

d = blade spacing along cascade axis
i = unit imaginary number with regard to space
j = unit imaginary number with respect to time, $ij \neq -1$
k = reduced frequency, $\omega d/U$
l = cavity length
M = transfer matrix
N = steady velocity normal to cascade axis
$\tilde{N}_1$ = unsteady upstream perturbation velocity normal to cascade
$\tilde{N}_2$ = unsteady downstream perturbation velocity normal to cascade
$\tilde{P}_1$ = residuary pressure at far upstream
$\tilde{P}_2$ = residuary pressure at far downstream
S = abscissa of cavity terminus in the ζ-plane
T_1 = steady upstream velocity tangential to cascade axis
T_2 = steady downstream velocity tangential to cascade axis
$\tilde{T}_2$ = unsteady downstream perturbation velocity tangential to cascade axis
u = perturbation velocity in the x-direction
U = steady upstream velocity in the x-direction
$\tilde{u}$ = unsteady part of perturbation velocity in the x-direction
v = perturbation velocity in the y-direction
$\tilde{v}$ = unsteady part of perturbation velocity in the y-direction
w = complex perturbation velocity ($w = u - iv$)
$\tilde{w}$ = unsteady part of complex perturbation velocity ($\tilde{w} = \tilde{u} - i\tilde{v}$)
z = complex variable in the physical plane, $z = x + iy$
α = steady angle of attack at upstream infinity
γ = stagger angle of cascade axis
ρ = fluid density
σ = cavitation number
ω = angular velocity of oscillation
ζ = complex variable in the transformed plane, $\zeta = \xi + i\eta$

Subscripts

1, 2 = conditions at upstream and downstream infinity, respectively
c = conditions on the cavity surface
s = steady solutions

Superscripts

$\sim$ = unsteady part

where $\tilde{w} = \tilde{u} - i\tilde{v}$, one can establish the boundary conditions for $\tilde{u}$ and $\tilde{v}$ in the ζ-plane as shown in Fig. 3:

$$\tilde{v} = 0, \ \xi < 0, \ \eta = 0$$

$$\tilde{u} = \tilde{g}e^{-jkx(\xi)/d}, \ 0 < \xi < s, \ \eta = 0$$

$$\tilde{v} = 0, \ \xi > s, \ \eta = 0$$

where

$$\frac{x(\xi)}{d} = \frac{1}{\pi}\left\{\frac{\cos\gamma}{2}\ \ln\left(1 - \frac{\xi}{\sqrt{d}}\sin\gamma + \xi^2/d\right)\right.$$
$$\left. - \sin\gamma\ \tan^{-1}\frac{-\xi\cos\gamma}{\sqrt{d} - \xi\sin\gamma}\right\}. \qquad (3)$$

The solution of this mixed-type Hilbert boundary value problem is readily found to be [10, 11]

$$\tilde{w}(\zeta) = \frac{\tilde{g}}{\pi\sqrt{\zeta(\zeta - S)}}\int_0^S \sqrt{\xi(S-\xi)}\, e^{-jkx(\xi)/d}\,\frac{d\xi}{\xi - \zeta}$$
$$+ \frac{A\zeta + B}{\sqrt{\zeta(\zeta - S)}}. \qquad (4)$$

Here, $\tilde{g}$, A, and B are constants (real in space, complex in time) to be determined shortly. S is given from equation (3) by substituting $\xi = S$ and $x(\xi) = l$. The following three conditions are available for the determination of $\tilde{g}$, A, and B:

(i) At upstream infinity, i.e., as $\zeta \rightarrow \zeta_1$,

$$\tilde{w} = \tilde{N}_1\cos\gamma + i\tilde{N}_1\sin\gamma$$

Equation (4) then gives

$$\frac{\tilde{g}}{\pi\sqrt{\zeta_1(\zeta_1 - S)}}\int_0^S \sqrt{\xi(S-\xi)}\, e^{-jkx(\xi)/d}\,\frac{d\xi}{\xi - \zeta_1}$$
$$+ \frac{A\zeta_1 + B}{\sqrt{\zeta_1(\zeta_1 - S)}} = \tilde{N}_1(\cos\gamma + i\sin\gamma). \qquad (5)$$

(ii) At downstream infinity, i.e., as $|\zeta| \rightarrow \infty$,

$$\tilde{w} = \tilde{N}_2\cos\gamma + \tilde{T}_2\sin\gamma + i(\tilde{N}_2\sin\gamma - \tilde{T}_2\cos\gamma)$$

Equation (4) thus gives

$$A = \tilde{N}_2\cos\gamma + \tilde{T}_2\sin\gamma + i(\tilde{N}_2\sin\gamma - \tilde{T}_2\cos\gamma). \qquad (6)$$

(iii) Also at far downstream, the flow should be parallel to the vanes. Therefore

$$\tilde{N}_2\sin\gamma = \tilde{T}_2\cos\gamma. \qquad (7)$$

The foregoing three conditions completely determine $\tilde{g}$, A, and B, and they can be written

$$A = \tilde{N}_2/\cos\gamma$$
$$\tilde{g} = g_1\tilde{N}_1 + g_2\tilde{N}_2 \qquad (8)$$
$$B = \sqrt{d}\,(B_1\tilde{N}_1 + B_2\tilde{N}_2)$$

where g_1, g_2, B_1, B_2 are frequency-dependent constants.

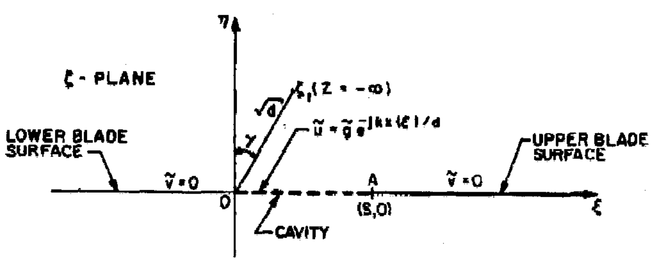

Fig. 3 Transformed plane with linearized boundary conditions

We now proceed to find the relation between (P_1, P_2) and (N_1, N_2). As $x \rightarrow -\infty$, the Euler's equation of motion gives

$$P \rightarrow -j\omega\rho\tilde{u}_1\,xe^{j\omega t} + \tilde{P}_1 e^{j\omega t} + P_{1s} \qquad (9)$$

where P_{1s} is the steady mean pressure at upstream infinity, and $\tilde{u}_1 = \tilde{N}_1\cos\gamma$. Likewise, as $x \rightarrow +\infty$, one may write

$$P \rightarrow -j\omega\rho\tilde{u}_2\,xe^{j\omega t} + \tilde{P}_2 e^{j\omega t} + P_{2s} \qquad (10)$$

where $\tilde{u}_2 = \tilde{N}_2\cos\gamma + \tilde{T}_2\sin\gamma$. Also, writing the x-velocity by $u_s + \tilde{u}\,e^{j\omega t}$ yields the following linearized equation of motion:

$$\frac{\partial}{\partial t}(u_s + u\,e^{j\omega t}) + \frac{\partial}{\partial x}\left(\frac{1}{2}u_s^2 + u_s\,u\,e^{j\omega t}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial x}.$$

Extracting the unsteady part only, one may write

$$j\omega\,\tilde{u} + \frac{\partial}{\partial x}(u_s\tilde{u}) = -\frac{1}{\rho}\frac{\partial\tilde{P}}{\partial x}.$$

Integrating this from $(-\infty, 0)$ to $(0^+, 0^+)$, one arrives at

$$U(\sqrt{1+\sigma}\,\tilde{g} - \tilde{N}_1\cos\gamma) - j\omega\int_{-\infty}^0 x\,\frac{d\tilde{u}(x,0)}{dx}\,dx$$
$$= -\frac{1}{\rho}(\tilde{P}_c - \tilde{P}_1). \qquad (11)$$

Equation (9) has been applied via the integration by part in the foregoing. Here, $\tilde{P}_c$ is the oscillatory pressure on the cavity and σ is the cavitation number defined by

$$\sigma = \frac{P_{1s} - P_{cs}}{{}^1\!/_2\rho U^2} \qquad (12)$$

where P_{cs} is the steady mean pressure on the cavity.

Similarly, the upstream infinity and the downstream infinity can be related by

$$U[(1 - \tan\alpha\tan\gamma)\tilde{N}_2/\cos\gamma - \tilde{N}_1\cos\gamma]$$
$$-j\omega\int_{-\infty}^0 x\,\frac{d\tilde{u}(x,0)}{dx}\,dx - j\omega\int_0^\infty x\,\frac{d\tilde{u}(x,0)}{dx}\,dx$$
$$= -\frac{1}{\rho}(\tilde{P}_2 - \tilde{P}_1). \qquad (13)$$

It can be shown that we may put

$$\int_{-\infty}^0 x\,\frac{d\tilde{u}(x,0)}{dx}\,dx = d(G_1\tilde{N}_1 + G_2\tilde{N}_2) \qquad (14)$$

$$\int_0^\infty x\,\frac{d\tilde{u}(x,0)}{dx}\,dx = d(H_1\tilde{N}_1 + H_2\tilde{N}_2) \qquad (15)$$

where G_1, G_2, H_1, and H_2 are frequency-dependent constants. Using equations (8), (14), and (15), we can rewrite equations (11) and (13) in the form

$$\vec{P} = M\vec{N} + \vec{P}_c \qquad (16)$$

where

$$\vec{P} = \frac{1}{\rho U^2}\begin{pmatrix}\tilde{P}_1\\ \tilde{P}_2\end{pmatrix}, \vec{N} = \frac{1}{U}\begin{pmatrix}\tilde{N}_1\\ \tilde{N}_2\end{pmatrix}, \vec{P}_c = \frac{\tilde{P}_c}{\rho U^2}\begin{pmatrix}1\\ 1\end{pmatrix},$$

and

$$M = \begin{bmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}$$

$$
= \begin{bmatrix} g_1\sqrt{1+\sigma} - \cos\gamma - jkG_1 & g_2\sqrt{1+\sigma} - jkG_2 \\ g_1\sqrt{1+\sigma} + jkH_1 & g_2\sqrt{1+\sigma} + jkH_2 - \dfrac{1 - \tan\alpha\tan\gamma}{\cos\gamma} \end{bmatrix} \cdot \tag{17}
$$

Thus we have obtained the relation between $(\tilde{P}_1, \tilde{P}_2)$ and $(\tilde{N}_1, \tilde{N}_2)$. The matrix M is a transfer matrix between these two sets of variables. Some numerical values of M are shown in Table 1 for $\alpha = 5$ deg and $\gamma = 75$ deg. The cavitation number σ was obtained from steady-state solution.

Specific Example

As a simple demonstration of how to apply this matrix M to a practical problem, it is interesting to consider an exemplary situation shown in Fig. 4. Here, we have an unsteady cavitating axial inducer pump at the end of a feed pipeline that is connected to a very large reservoir. For simplicity, let us take $P_e = 0$. Assume L_1/d, $L_2/d >> 1$ where d is the vane spacing of the pump. The pressures on the reservoir surface and the discharge section are given as in the figure. We would like to find the fluctuating normal velocity components (N_1, N_2) in terms of the given pressures.

The Bernoulli's equation between the reservoir surface and the point A can be written

$$
P_{0s} + P_0 e^{j\omega t} + \rho g y_0 = P_{As} + P_A e^{j\omega t} + \frac{\rho}{2}(N + N_1 e^{j\omega t})^2 + \rho \left.\frac{\partial \phi}{\partial t}\right|_A .
$$

Here, ϕ is the velocity potential. Writing $\phi = \phi_s + \phi\, e^{j\omega t}$, the unsteady part of the foregoing Bernoulli's equation yields

$$
P_0 = P_A + \rho N N_1 + j\omega\rho\, \phi_A \tag{18}
$$

P_A consists of both the residuary part P_1 and the inertial part. To separate them out, let us consider the equivalent cascade problem illustrated in Fig. 5. Here, since L_1 is large, we may approximate by equation (9)

$$
P_A(t) \cong j\omega\rho\, N_1 \cos\gamma\, (L_1/\cos\gamma)\, e^{j\omega t} + P_1 e^{j\omega t} + P_{As}
$$

so that we may identify

$$
P_A = j\omega\rho\, L_1 N_1 + P_1
$$

Substituting this into equation (18) results in

Table 1 Numerical values of M for $\alpha = 5$ deg, $\gamma = 75$ deg

$l/d = 0.01$ ($\sigma = 12.275$)

k	M_{11}	M_{12}	M_{21}	M_{22}
0	4.308×10^{2}	-5.365×10^{2}	4.310×10^{2}	-5.391×10^{2}
0.1	4.308×10^{2} $+j(1.799 \times 10^{-1})$	-5.365×10^{2} $-j(1.788 \times 10^{-1})$	4.310×10^{2} $+j(5.629)$	-5.391×10^{2} $-j(7.979)$
0.5	4.308×10^{2} $+j(8.977 \times 10^{-1})$	-5.363×10^{2} $-j(8.940 \times 10^{-1})$	4.310×10^{2} $+j(2.814 \times 10)$	-5.390×10^{2} $-j(3.989 \times 10)$
0.7	4.308×10^{2} $+j(1.260)$	-5.365×10^{2} $-j(1.252)$	4.310×10^{2} $+j(3.940 \times 10)$	-5.390×10^{2} $-j(5.585 \times 10)$
1.0	4.308×10^{2} $+j(1.799)$	-5.365×10^{2} $-j(1.788)$	4.309×10^{2} $+j(5.629 \times 10)$	-5.389×10^{2} $-j(7.970 \times 10)$

$l/d = 0.1$ ($\sigma = 2.137$)

k	M_{11}	M_{12}	M_{21}	M_{22}
0	1.779×10	-3.267×10	1.805×10	-3.528×10
0.1	1.779×10 $+j(6.080 \times 10^{-2})$	-3.267×10 $-j(6.258 \times 10^{-2})$	1.804×10 $+j(3.170)$	-3.526×10 $-j(8.564)$
0.5	1.777×10 $+j(3.039 \times 10^{-1})$	-3.265×10 $-j(3.128 \times 10^{-1})$	1.787×10 $+j(1.585 \times 10)$	-3.496×10 $-j(4.283 \times 10)$
0.7	1.777×10 $+j(4.255 \times 10^{-1})$	-3.265×10 $-j(4.380 \times 10^{-1})$	1.771×10 $+j(2.219 \times 10)$	-3.467×10 $-j(5.996 \times 10)$
1.0	1.777×10 $+j(6.079 \times 10^{-1})$	-3.265×10 $-j(6.258 \times 10^{-1})$	1.737×10 $+j(3.169 \times 10)$	-3.406×10 $-j(8.565 \times 10)$

$l/d = 0.5$ ($\sigma = 0.559$)

k	M_{11}	M_{12}	M_{21}	M_{22}
0	1.209	-4.575	1.468	-7.177
0.1	1.209 $+j(1.539 \times 10^{-2})$	-4.575 $-j(1.746 \times 10^{-2})$	1.446 $+j(7.381 \times 10^{-1})$	-7.109 $-j(1.138 \times 10)$
0.5	1.207 $+j(7.702 \times 10^{-2})$	-4.567 $-j(8.748 \times 10^{-2})$	9.188×10^{-1} $+j(3.687)$	-5.465 $-j(5.689 \times 10)$
0.7	1.204 $+j(1.079 \times 10^{-1})$	-4.560 $-j(1.227 \times 10^{-1})$	3.906×10^{-1} $+j(5.155)$	-3.819 $-j(7.962 \times 10)$
1.0	1.199 $+j(1.544 \times 10^{-1})$	-4.543 $-j(1.761 \times 10^{-1})$	-7.345×10^{-1} $+j(7.348)$	-3.136×10^{-1} $-j(1.137 \times 10^{2})$

$l/d = 1.0$ ($\sigma = 0.111$)

k	M_{11}	M_{12}	M_{21}	M_{22}
0	7.066×10^{-2}	-5.426×10^{-1}	1.882×10^{-1}	-3.145
0.1	-7.067×10^{-2} $+j(1.041 \times 10^{-2})$	-5.425×10^{-1} $-j(6.660 \times 10^{-3})$	1.660×10^{-1} $-j(2.348)$	-3.081 $-j(4.646)$
0.5	-7.109×10^{-2} $+j(5.209 \times 10^{-2})$	5.414×10^{-1} $-j(3.336 \times 10^{-2})$	3.676×10^{-1} $-j(1.175 \times 10)$	-1.542 $-j(2.321 \times 10)$
0.7	-7.150×10^{-2} $+j(7.295 \times 10^{-2})$	5.402×10^{-1} $-j(4.679 \times 10^{-2})$	-9.038×10^{-1} $-j(1.646 \times 10)$	4.025×10 $-j(3.247 \times 10)$
1.0	-7.239×10^{-2} $+j(1.043 \times 10^{-1})$	-5.376×10^{-1} $-j(6.712 \times 10^{-2})$	-2.052 $-j(2.353 \times 10)$	3.315 $-j(4.631 \times 10)$

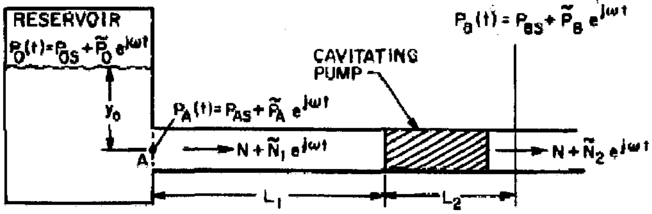

Fig. 4 Exemplary model for cavitating inducer pump in a long channel connected to a large reservoir

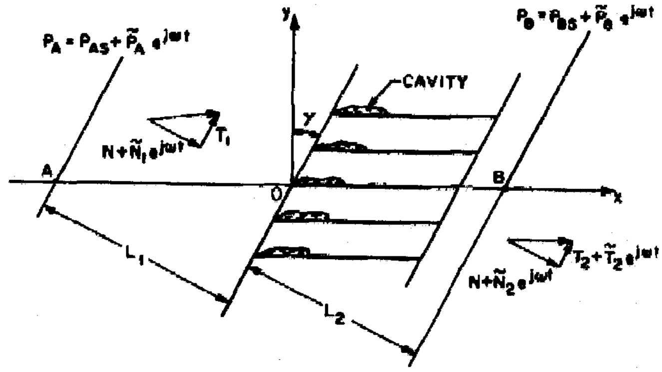

Fig. 5 Equivalent cascade flow for cavitating inducer pump of Fig. 4.

$$P_1 = P_0 - \rho N N_1 - (j\omega\rho\, L_1\, N_1 + j\omega\rho\, \phi_A).$$

Defining the "effective" length of the feed line L_0 by $j\omega\, L_0 N_1 \equiv j\omega\, L_1 N_1 + j\omega\, \phi_A$, one obtains

$$P_1 = P_0 - (\rho N + j\omega\, L_0\, \rho) N_1. \tag{19}$$

Similarly, since L_2 is large, we may approximate

$$P_2 \cong P_B + j\omega\rho\, L_2 N_2/\cos^2\gamma. \tag{20}$$

Substituting equations (19) and (20) into equation (16) and solving for N and noting that $U = N\cos\alpha/\cos(\alpha+\gamma)$, we finally obtain

$$N_1/N = \frac{P_0\left(M_{22} - jk\,\dfrac{L_2}{d\cos^2\gamma}\right) - P_B M_{12}}{\rho N^2 D} \tag{21}$$

$$N_2/N = \frac{-P_0 M_{21} + P_B(M_{11} + \cos(\alpha+\gamma)/\cos\alpha + jk\, L_0/d)}{\rho N^2 D} \tag{22}$$

where

$$D = \left[1 + \frac{\cos\alpha}{\cos(\alpha+\gamma)}(M_{11} + jk\, L_0/d)\right]\left[M_{22} - jk\,\frac{L_2}{d\cos^2\gamma}\right] - M_{12}M_{21}\,\frac{\cos\alpha}{\cos(\alpha+\gamma)}. \tag{23}$$

Thus N_1 and N_2 are completely determined in terms of known pressures through the matrix M. To elucidate the situation more clearly, let us consider this time a "slug" flow and compare the result thus obtained with those of the foregoing case. In a slug flow, the whole flow moves in phase, and also we should have $P_1 = P_2$ and $N_1 = N_2 = N$, say. In this case, equations (19) and (20) yield

$$\frac{\tilde{N}}{N} = \frac{\tilde{P}_0 - \tilde{P}_B}{\rho N^2}\,\frac{1}{\left[1 + jk\,\dfrac{\cos\alpha}{\cos(\alpha+\gamma)}\left(L_0/d + \dfrac{L_2}{d\cos^2\gamma}\right)\right]}. \tag{24}$$

Here, L_0 is defined by $j\omega\, L_0 N \equiv j\omega\, L_1 N + j\omega\, \phi_A$. Comparing equation (24) with equations (21) and (22), it is evident that the matrix M can be thought of as being a measure of the residuary dynamic effect due to the blade-unsteady cavity system in addition to the overall general oscillation of the flow as a slug.

Numerical values for these two different types of flow are compared in Table 2 for $l/d = 0.5$ and $l/d = 1.0$. Here, we have taken $P_B = 0$, $P_0 = 2$ percent of the atmospheric pressure, $L_0/d = 50$, $L_2/d = 20$, $\rho = 1$ gr/cm^3 ($= 1.939$ slug/ft^3), $N = 10.44$ m/s ($= 30$ ft/s), $\alpha = 5$ deg and $\gamma = 75$ deg. It is seen from these calculations that the values of N_1/N and N_2/N for the cavitating flow case are appreciably different from those of N/N corresponding to a slug flow.

In the presence of vanes but without cavity, we will still have $N_1 = N_2 = N$, but in general P_1 and P_2 will be related by $P_1 - P_2 = NN\tan^2\gamma$ through the unsteady Bernoulli's equation.

Table 2 Numerical values of N_1/N, N_2/N, and N/N for the exemplary problem ($\alpha = 5$ deg, $\gamma = 75$ deg, $P_B = 0$, $P_0 = 2$ percent of 14.7 psia, $L_0/d = 50$, $L_2/d = 20$, $\rho = 1$ gr/cm^3, $N = 10.44$ m/s)

$l/d = 0.5$ ($\sigma = 0.559$)			
k	N_1/N	N_2/N	N/N
0	8.14×10^{-2}	1.67×10^{-2}	2.43×10^{-2}
0.1	-1.43×10^{-4} $+j(1.77\times10^{-3})$	9.60×10^{-5} $+j(1.60\times10^{-4})$	6.77×10^{-5} $+j(1.28\times10^{-3})$
0.5	-8.51×10^{-6} $+j(3.76\times10^{-4})$	2.94×10^{-6} $+j(2.05\times10^{-5})$	2.72×10^{-6} $+j(2.57\times10^{-4})$
0.7	-4.36×10^{-6} $+j(2.69\times10^{-4})$	2.85×10^{-7} $+j(1.45\times10^{-5})$	1.39×10^{-6} $+j(1.83\times10^{-4})$
1.0	-2.13×10^{-6} $+j(1.88\times10^{-4})$	-1.15×10^{-6} $+j(1.01\times10^{-5})$	6.79×10^{-7} $+j(1.28\times10^{-4})$
$l/d = 1.0$ ($\sigma = 0.111$)			
k	N_1/N	N_2/N	N/N
0	1.92×10^{-2}	1.15×10^{-3}	2.43×10^{-2}
0.1	1.14×10^{-4} $+j(1.85\times10^{-3})$	2.34×10^{-4} $-j(5.19\times10^{-4})$	6.77×10^{-5} $+j(1.28\times10^{-3})$
0.5	4.19×10^{-6} $+j(3.77\times10^{-4})$	2.10×10^{-7} $-j(1.26\times10^{-4})$	2.72×10^{-6} $+j(2.57\times10^{-4})$
0.7	2.13×10^{-6} $+j(2.69\times10^{-4})$	-5.70×10^{-6} $-j(9.05\times10^{-5})$	1.39×10^{-6} $+j(1.83\times10^{-4})$
1.0	1.05×10^{-6} $+j(1.88\times10^{-4})$	-8.88×10^{-6} $-j(6.30\times10^{-5})$	6.79×10^{-7} $+j(1.28\times10^{-4})$

In this case, the quantity inside the bracket of equation (24) will contain an additional term $\tan^2 \gamma$.

Concluding Remarks

In the present formulation, a type of cavitation compliance similar to that of reference [7] will have the form

$$K^* = -\frac{1}{d^2}\frac{\partial V^*/\partial t}{\partial \sigma^*/\partial t},$$

where V^* is the cavity volume per unit depth of the plane and

$$\sigma^* = \frac{P_1(t) - \tilde{P}_c(t)}{\frac{1}{2}\rho U^2}, \; P_1(t) = P_{1s} + \tilde{P}_1 e^{j\omega t}$$

and $P_c(t) = P_{cs} + \tilde{P}_c e^{j\omega t}$. This allows us to express

$$K^* = -\frac{1}{d^2}\frac{d(\tilde{N}_2 - \tilde{N}_1)e^{j\omega t}}{j\omega(\tilde{P}_1 - \tilde{P}_c)e^{j\omega t}/\frac{1}{2}\rho U^2}$$

This may be rearranged in a form

$$K^* = -\frac{\tilde{N}_2/N - \tilde{N}_1/N}{2jk(M_{11}\tilde{N}_1/N + M_{12}\tilde{N}_2/N)}$$

K^* thus obtained clearly gives a complex value, that is, the additional dynamic response beyond the purely inertial oscillation of a slug flow has both resistive and reactive terms. It also shows a strong dependence on frequency.

Experimental data for cavitation compliance were collected by Vaage, Fidler, and Zehnle [12]. However, these experimental values are by no means direct measurement results, but are inferred from rather complex dynamic models into which experimental observations were substituted. More direct test measurement data are clearly desired in this area.

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DISCUSSION

B. Lakshminarayana[3]

The authors are to be commended for the simple approach to one of the most difficult and least understood phenomena in fluid machinery. With a view toward practical application, I would like the authors to clarify some of the assumptions made.

1 Our experience (based on airflow tests at Penn State and model tests with water by NASA Lewis Research Center) indicates that the flow in these inducer passages is highly three dimensional, viscous, and rotational. Substantial pressure and velocity gradients exist in all directions, more so in the radial direction. The secondary and boundary layer flows dominate over the "primary" inviscid flow. The steady flow calculations based on inviscid cascade theories are nowhere near the measured properties. Can the authors comment on whether their theory holds good under these circumstances or can we treat the steady flow as viscid and rotational and unsteady part as irrotational and inviscid, assuming of course that the unsteady stresses are small compared to unsteady inertial and pressure forces?

2 The authors assume that the cavity depth is infinitely thin, whereas in actual practice the ratio of cavity depth to blade spacing can be appreciable. It would be useful if the authors could explain, qualitatively, what effect this has on their transfer matrix.

3 Presence of two-phase medium in the passage (vapor-liquid mixture), as well as vapor bubbles are known to cause the instability in the operation of the liquid rocket engines. The authors have studied the latter effect. Recent work by Kolesnikov and Kinelev (*Izvestia Vuz, Aviatsionnaya technika*, Vol. 16, No. 4, 1973, pp. 87–92) indicate that the former effect may be important, since even a slight amount of vapor phase in the flow medium is accompanied by a marked reduction in speed of sound. The pressure and velocity perturbations at the exit are transmitted to the inlet. Can the authors explain which of these two phenomena is more important in the study of instability?

4 The authors assume that the blades are infinitely long. Can the authors explain whether or not the transfer matrix would depend on the (cavity length/chord length); if so, how? Could this be done by defining the reduced frequency based on chord length (as is done by most of turbomachinery unsteady aerodynamists) instead of blade spacing?

The authors have provided an important groundwork for future studies on dynamic inducer cavitation behavior. The data from the experimental program underway at Cal Tech should provide the check on the assumptions made in this as well as other theories proposed for the pump instability.

C. C. S. Song[4]

I would like to commend the authors for their work attempting to explain the complicated phenomenon of feed-line cavitating-pump instability. It appears that the use of unsteady cavitating cascade theory is a right approach to partially explain the "compliance" effect of the cavitating pump. Perhaps a combination of unsteady cavity flow theory and the theory of hydraulic transients is needed to complete the solution.

Following are some of the questions I believe need clarification.

1 In this model authors considered semi-infinitely long cascade blades. How realistic is the model in representing the actual pump blades? It is well known that the existence of a free vortex sheet behind a foil is a primary factor in determining the unsteady force.

2 In dealing with an unsteady cavity flow problem, previous

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investigators often used an additional constraint condition, such as the constant cavity volume condition and the cavity closure condition to render the solution unique. Is such a condition required for this model?

3 Closely related with question 2 are the questions of pressure at infinity and the convergence of equations (14) and (15). Did the authors encounter any problem in performing the integrations appearing in these equations? Also, is the pressure at infinity as given by equation (9) bounded at infinity?

4 There is no indication given in the paper as to the method of computing $\tilde{\phi}_A$ appearing in equation (18). Is this function bounded at infinity?

5 In the "specific example" authors extended the potential flow solution to the entrance of the pipe, point A in Fig. 4. In the real case, however, the potential flow is applicable only near the pump. When the feed line L_1 is very long, as in the case considered by the authors, it appears more realistic to regard the flow in L_1 as that of one-dimensional transient flow in an elastic pipe. What is the justification for the apparent neglect of the transient effect?

Authors' Closure

The authors appreciate the important comments of the discussers each of whose own work is well known in the fields of cavitation and turbomachines. As a general remark, the present work can only be thought of as a qualitative guide to one aspect of the unsteady cavitating flow in real pumps. We would certainly agree with Dr. Lakshminarayana that viscous effects are important in inducer pumps but insofar as the effects of *attached blade cavitation* are concerned, the main trends of the steady and unsteady flow are not, we believe, dominated by viscosity. As an example of this trend, reasonable estimates of the "breakdown" cavitation number can be made with inviscid cavity flow cascade analysis. But of course it would be highly desirable to make internal flow measurements in cavitating inducers and we would hope that this type of observation may eventually become available. Apart from this, the methods of analysis are those of fully linearized free streamline theory and the resulting performance coefficients are subject to the limitations inherent in this theory. But as reference to the work of Furuya [13][4] shows, the linear theory is often a reasonable qualitative guide for steady flow even under conditions where the cavity thickness is not "thin" but the other conditions of the linearization are satisfied. We would expect a similar result for the present unsteady problem.

We appreciate the reference for the Russian literature; we are aware of the possible effects of free stream bubbles and the effect on pump dynamics. Indeed, the recent analyses by C. Brennen [14] suggest that this effect on the cavitation compliance can be significant, and further research in this area may be necessary.

[4]Numbers in brackets designate Additional References at end of closure.

Both Drs. Song and Lakshminarayana express concern about the infinite solidity of the partially cavitating cascade. The cascade sections of representative inducer pumps are often large compared to unity and the additional effects brought about by cavitation in steady flow can often be treated as if this solidity was infinite. The results of such an approximation would seem to provide a useful guide for the preliminary exploration of unsteady effect also. For this to apply to a plausible physical system, it seems reasonable (lacking a full analysis) that the reduced frequency based on blade spacing should not be too high. It is not difficult in principle to construct a solution for the finite solidity case (including all trailing vortex wakes) although the details of the analysis become much more tedious. The authors are, however, deeply interested in this "axial gust" problem and hope to present additional results in the near future to clarify this admittedly important effect.

In reply to Dr. Song's second comment, the fluctuating cavity volume is in fact the key element in the problem as it is only this that permits the up- and downstream normal velocity components to differ. But it is regrettably not very clear from the text just what the conditions on the cavity and its termination are; the cavity termination point, s of Fig. 3, is assumed to remain fixed during the oscillation and the boundary of the cavity moves to provide the net source strength of the fluctuating volume. Other termination models are certainly possible but the present one which is similar to Leehey's [15] was adopted for simplicity.

As to the remaining questions, convergence can be shown of integrals in equations (14) and (15) in a straightforward manner by performing the integrations in the ζ-plane. The total pressure at infinity as given by equation (9) is not bounded and need not be bounded as we allow the oscillation of the flow. However, the *residuary* pressure there is bounded.

The term $\tilde{\phi}_A$ is to be determined from a separate potential flow analysis of the inflow into the pipe; it is of the order of the reference velocity times a channel height. Finally, in commenting upon the use of the present results we could imagine that the effect of inserting the "pump-cascade" system into any hydraulic network could be accounted for by use of the relations of equation (16). This should be so because these "residual" pressure effects are in effect determined by the flow within a few blade spacings only. Thus, the feed line L_1 of the example could well have been replaced by an elastic one and the equations governing wave propagation in this pipe would be terminated at the pump site with equation (16). The example of the text was only to show in a simple case how these separate pump and feed line effects could be coupled.

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